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Context-dependent statements and consequences for the mathematical education of engineering students

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ABSTRACT: Context-dependent or incomplete statements often occur in everyday life. Many people use them intuitively without thinking about the possible consequences. Unfortunately, such statements can cause misunderstanding or even manipulate people. Some examples are mentioned in this article. Therefore, it is important to explain the context in a very responsible manner considering also the pre-knowledge of the audience. In mathematics and mathematically based sciences, incomplete concepts, models, assumptions, statements, and so on, can lead to contradictions and in applications to disastrous consequences. This article illustrates this by instructive examples. Hence, engineering students should learn to think logically, mathematically and rationally. As an important side effect of this basic science education, they should be able to discover weaknesses and contradictions in argumentation or to expose manipulation, half-truths and swindle in professional and everyday life.

INTRODUCTION

In everyday life people are accustomed to make incomplete and redundant statements. Some prerequisites are involved implicitly. Often, the audience knows them and so it is not necessary to stress them. But, sometimes misunderstandings result nevertheless. On the other hand, half-truths and fillers loaded with emotions occur in communication to show the involvement or to influence others in a calculated way. In principle, this is also the case in science and scientific education, hopefully, except the parts where (absolute) truth is necessary as in mathematics or the mathematically based sciences. Here, incomplete statements (in theorems, lemmas or propositions) are incorrect and lead to contradictions or wrong conclusions. This can be revealed by showing counter examples.

Concepts, methods and mathematical models in applications are as context-dependent as statements are. For instance, concepts can have the same name, but mean different things dependent on the field of application. If the prerequisites are not observed, in the worst case, the consequences can be disastrous. This is demonstrated by instructive examples. Moreover, proposals for the mathematical education are derived.

Thus, engineering students have to learn mathematical thinking. Part of it is thinking logical, considering also exceptions, learning about the variety of possibilities, detecting weaknesses in arguments, estimating results and the progress of small errors, etc. Perhaps, this general ability is more important than the right selection of mathematical subjects regarding the study course and possible applications in later professional life.

Mathematical education should train exact, correct and strict thinking, as well as fostering creativity and discovery. This leads to a better and deeper understanding of basics and generally to a critical view in everyday life. One prerequisite of detecting context-dependence is mathematical competence [1].

EXAMPLES IN MATHEMATICS

Critical thinking is indispensable in mathematics. Proofs have to be absolutely safe and reliable. All cases have to be considered, also the exceptions. Flaws in reasoning must be detected and eliminated.

In mathematics, statements theorems, lemmas, propositions and corollaries contain a list of assumptions and a list of assertions. The proofs show that the assertions follow by already proven statements and by logical conclusions using the assumptions. The teachers and, hence, the students often try to simplify these. They omit some or all of the assumptions and appeal to the plausibility of statements, which are often only fulfilled for simple or typical cases. Since proofs are omitted in the most cases too, the falsity of the statements is normally not noticed. The falseness can be verified by constructing *counterexamples* [2].

Often, formulas or theorems are given incompletely, omitting some assumptions as already mentioned. Starting with a simple algebraic formula.

- Proposition 1a: $\sqrt{x^2} = x$.

The proposition can be supported by the argument that calculating powers and corresponding roots are inverse operations, which neutralise each other if applied one after the other. If a student checks this proposition using natural or non-negative real numbers it is confirmed. But, if $x = -3$ is put into the left-hand side of the equation the right-hand side produces $x = +3$. This is a counterexample. Here, tacitly the convention is used that the square root supplies a nonnegative value. Hence, Proposition 1 is only true for $x \geq 0$, which should be added to the proposition. But, the result can be extended to arbitrary real numbers by use of the absolute value.

- Proposition 1b: $\sqrt{x^2} = |x|$ for all real *x*.

This is a correct proposition. Mathematics looks for generalisation. What is the meaning of the left-hand side if complex numbers *x* are admitted? The square root has then two values, namely *x* and -*x*. If one wants to express this again by an equation, one must write:

- Proposition 1c: $\sqrt{x^2} = \{x, -x\}$.

Hence, the intended range of numbers *x* must be known. Another simple equality is:

- Proposition 2: $\dfrac{x^2 - 1}{x - 1} = x + 1$.

A proof could use the binomial formula $(x+1)\cdot(x-1) = x^2 - 1$, which is true for all real and even for all complex numbers *x*. But the left-hand side of the equation is not defined for $x = 1$. Hence, Proposition 1 holds only for $x \neq 1$. This becomes evident if the binomial formula is used in the nominator of the fraction:

$$\frac{x^2 - 1}{x - 1} = \frac{(x+1)\cdot(x-1)}{x - 1} = x + 1 \quad (x \neq 1) ,$$

since, only nonzero factors in fractions can be cancelled. Hence, the admissible range for *x* must be given in the proposition. The next example shows that one has to observe the ranges of definition on both sides of an equation.

- Proposition 3: $\sqrt{x-2}\cdot\sqrt{x+2} = \sqrt{x^2 - 4}$.

This seems to be clear because of $(x-2)\cdot(x+2) = x^2 - 1$ and a root law. But, for real numbers, roots are only defined for non-negative radicands. Starting with the left-hand side of Proposition 3, the first root needs $x \geq 2$ and the second root $x \geq -2$. Hence, the product is defined only for $x \geq 2$. But, the right-hand side is defined for a larger range, namely for $|x| \geq 2$, the union of the first given ranges. Nevertheless, the equality holds only for $x \geq 2$. This should be added to Proposition 3. The next proposition reflects again a root problem.

- Proposition 4: $\sqrt[6]{(-1)^2} = \sqrt[3]{-1} = -1$ and $\sqrt[6]{1^2} = \sqrt[3]{1} = 1$.

Both calculations use root laws. Nevertheless, this leads to a contradiction because both sixth roots must have the same value because of $(-1)^2 = 1^2 = 1$. Consequently, the root laws have to be applied carefully. The conflict vanishes if the expressions are interpreted in the field of complex numbers. Here, $\sqrt[6]{1}$ has 6 different values, the both real numbers 1 and -1 and 4 complex numbers. Now, a theorem about functions is presented.

- Proposition 5: the function $F(x) = \dfrac{f_1(x)}{f_2(x)}$ has a vertical asymptote (or point of infinity) at $x = a$ if $f_2(a) = 0$.

This proposition is not correct in general [2]. It means that $\lim\limits_{x \to a \pm 0} |F(x)| = +\infty$. But, in the case $f_1(a) = 0$ this limit can also be finite. A simple counterexample is $f_1(x) = \sin x$, $f_2(x) = x$ and $a = 0$. It is well-known that:

$$\lim_{x\to 0} F(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

The discussion of Proposition 2 supplies the counterexample $f_1(x) = x^2 - 1$, $f_2(x) = x - 1$ and $a = 1$, which shows that Proposition 5 is not even true for rational functions $F(x)$, where nominators and denominators are polynomials [2]. Here, it is:

$$\lim_{x\to 1} F(x) = \lim_{x\to 1} (x+1) = 2.$$

Hence, Proposition 5 has to be supplemented by the assumption $f_1(a) \neq 0$. Further, f_1 and f_2 should be defined in a neighbourhood of $x = a$. Another theorem follows.

- Proposition 6: the real function f has a zero in the interval $[a,b]$, if $f(a) \cdot f(b) < 0$ holds.

This sounds plausible because the assumption means that f has different signs at the endpoints of the interval. This need not hold for functions which jump inside the interval. But, adding the continuity of f on $[a,b]$ Proposition 6 is correct. But, observe that f can have more than one zero and that f can also have zeros in the case that the assumption is violated (sufficient, but not necessary condition). The last example concerns the relation between exact and approximate calculations.

- Proposition 7: if x is a complex number with $|x| = 1$, then $|x^n| = 1$ holds for all natural n.

Proposition 7 is, indeed, true and can be proved. But, a computer check by numerical calculation will supply surprising results depending on the structure of particular computer numbers. The first observation is that there are small rounding errors, such that $|x^k| \neq 1$ will occur for some k. There are two possibilities. If $|x^k| < 1$, then $|x^n|$ will tend to zero. If $|x^k| > 1$, then $|x^n|$ will tend to infinity. This effect shows that numerical calculations can have disastrous consequences.

Concepts are often context-dependent. Mathematics is no exception here. Two examples are given below.

- *Solution of a problem*: classical solution, generalised solution. Each classical solution is a generalised solution, but not *vice versa*. In some fields, a lot of solution concepts are available. If some of these occur in a paper, they must get a name addition. If a solution is mentioned without the context, this can lead to misunderstandings. In MATLAB, for instance, the solution command for a linear system of equations supplies the so-called least-squares solution, which is a best approximation (or optimisation) concept. People who do not know this concept often assume that it is a classical solution. This misinterpretation can have serious consequences. A simple check can clarify the situation, but this needs some mathematical background.
- *Limit of a sequence*: classical limit, generalised limit, definition of *convergence*. Each classical limit is also a generalised limit. But, there are divergent sequences which have a generalised limit. Again, there are a lot of possibilities, mostly based on mean methods. The sequence with alternating members 0 and 1 is divergent in the classical sense. A generalised limit would supply the value 0.5, the mean value of 0 and 1. Just this value is given as the limit using MATLAB. People who do not know about this can derive wrong results.

A unique answer in these cases needs the context.

EXAMPLES IN SCIENCE

Modelling. Models are used in the natural sciences. Models simplify and neglect some aspects and in so far, models are incomplete. The selection of models depends on the context. An oscillator model is shown in Figure 1.

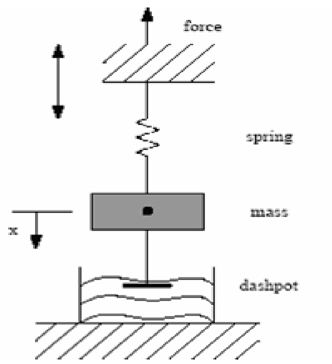

Figure 1: Harmonic oscillator with damping.

Is the oscillator approximately harmonic or not? If a usual spring is used the answer is *yes*. Can damping be neglected? Normally, it cannot. But, what kind of damping should be used, viscous damping or turbulent damping? It depends on the medium and on the velocity [3]. Good modelling is an art, and it has to observe the special context. Otherwise, completely wrong predictions from the model can occur.

The body mass index (BMI) modelling body constitution of humans with relation to health or beauty is analysed by the author elsewhere [4][5]. Here, the ratio of weight and the squared height of a person is taken as essential. But, is it really? It should be remembered that the Belgian mathematician and statistician Adolphe Quételet (1796-1874) proposed it in 1870 after inspecting the build of 5,738 Scottish soldiers, which were in no way representative of the earth's contemporary population. A further disadvantage of the BMI model is that proportional body structures can cause quite different BMI values [6].

By the way, there are other measures for beauty or health. The definition of these concepts is fuzzy. It depends again on the context, which features are the most important. One should reflect now on some special methods used to derive statements.

Experiments. The conditions underlying an experiment must be clearly described so another team can check the results of an experiment. If different results are derived, something must be wrong. Otherwise different teams can obtain different results, which are both true under the corresponding conditions. This is not acceptable in science.

Statistics. In this field random samples of objects are investigated to conclude to the total object set. A sample is naturally incomplete. Hence, estimations and probability statements with uncertainties can be obtained. First of all, the sample must be representative that is it has to represent the typical features. Second, the sample should be large enough to reduce the uncertainty.

Questionnaires. Again, often one only has a sample. Perhaps the quality of a study course has to be measured. Then, suitable questions have to be selected, and psychological aspects influence the answers.

News. In newspapers, scientific results are often presented incompletely. For instance, one can read about studies or investigations, which prove that drinking wine or green tea are good for the health while eating eggs is bad. Sometimes these statements also change to the contrary. The reader is influenced in his or her behaviour, and sometimes also insecure. The reader gets no or not enough information about restrictions. Surely the amount of consumed substances per day plays an important part for the statement. The reader can trust it or not. In both cases there can be disappointment afterwards.

EXAMPLES IN EVERY DAY LIFE

Often tacit conventions, which are not known by all people are used. Concerning *events* or public *talks* the speakers should also give the context of their theses. The audience should ask for the context if it is not given or not clear enough. Hopefully, there are serious and honest discussions, which avoid personal attacks or attempts for manipulation.

Rankings are useful or necessary in everyday life (job, classification, award, bonus, reward). But, there are uncertainties. Which criteria are chosen; what weights should criteria get; is there a clear interpretation of the criteria? Playing with criteria can change the ranking, possibly in an unsuspected manner.

Decisions are often difficult to find in a commission, because there are different interests. However, all facts must be tabled to ensure reasonable results. Perhaps, compromises are necessary. All facts should be considered and these should be balanced.

Quotations must be given completely and with the necessary context. Sometimes certain journalists and politicians try to take a statement out of context in order to blame or damage other persons.

The following theses have political aims. They seem to be clear, but a thorough discussion shows some vagueness or uncertainty. At the same time, they are part of certain ideologies.

- Thesis 1: If freedom and human rights are endangered, if necessary one should take up arms to defend these values.

Is this acceptable or prudent under all circumstances? And, further: where are these values becoming endangered? In their own country, in a country connected by alliance or in other regions of the world? Finally, how should the restriction *if necessary* be understood? Questions over questions are an important topic for long discussions.

- Thesis 2: Islam belongs to Germany.

What should this statement mean? If the meaning is that people with Islam religion are living in Germany, the answer is *yes*. If the meaning is, that Islam has shaped the German cultural tradition, the answer is *no*. If the meaning is, that Islam can contribute to German culture in future, the answer is probably *yes*. The next statement is a slogan arranged before the German election for the European parliament in 2014.

- Thesis 3: Only if you vote for Martin Schulz and the Social Democratic Party of Germany (SPD) can a German become the president of the EU commission.

This statement is wrong and irritating, but follows a political aim. Martin Schulz could also become the president if you vote for another party. At first, it sounds plausible because the only opposing candidate in Europe is not a German, namely Jean-Claude Juncker from Luxembourg although he speaks German very well. The message is that you should vote for the SPD in Germany. If enough people follow this message, then, the chance to get a German EU commission president is not bad under the condition that enough people in other European countries also vote for their social democratic party. Naturally, there is a dilemma for people who would not normally vote for the SPD, but who would prefer a German EU president. The hope is that these people will vote for the SPD from national motives. However, initiated people know that at the moment there is a power struggle between the European Parliament and the European Council of heads of government, which must accept (at least until now) the candidate and which can also propose a new candidate, for instance, a German with a name other than Martin Schulz. In the meantime, the European Parliament has won the power struggle and Juncker as the representative of the strongest fraction of conservative parties has become the president.

Concepts are often vague, fuzzy or have more than one meaning. Two related examples are mentioned here.

- *Norm.* This can mean, e.g. a) (behavioural) rule; b) (near) average level; c) guideline; d) standard; e) working performance; and f) metric, distance (from zero element) in mathematics.
- *Normal.* This can mean, e.g. a) corresponding to the norm; b) correct; c) proper; d) conventional; and e) usual.

It is difficult or also problematic to define what *normal* really means without giving a special context. Often people think that *normal* characterises the typical or the average. Indeed, many features in nature are normally distributed (Gaussian normal distribution with one symmetric peak at the expectation value). But in many cases, especially concerning modern trends, the average is in no way the normal (consider, e.g. distributions with more than one peak, see Figure 2).

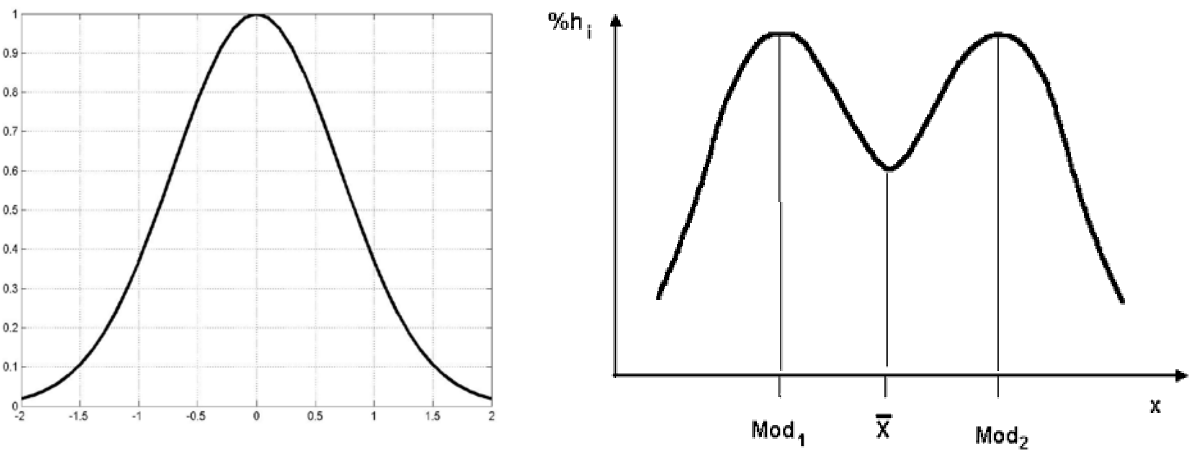

Figure 2: Distributions with one symmetric peak and two peaks.

The already mentioned Adolphe Quételet even undertook a study about the *average human* which caused vehement debates. There is an average of all human weights and all human heights at time t, but what should be the average human? Nevertheless, Quételet started the quantitative analysis of human and social qualities [6].

CONCLUSIONS

The essential findings of this topic are similar to these given by the author elsewhere [4][5].

- Critical thinking can be transferred from mathematics to science, profession and everyday life.
- Problem solving needs a given context. Vague statements should be avoided or completed if possible.
- A reasonable curriculum should not only supply problem-solving competencies in mathematics and engineering, but also in everyday life by making general processes more transparent.
- Mathematical educators should encourage young people to act not only emotionally to societal developments, but also rationally using mathematical or logical arguments.
- For everyday life it is more important in mathematics to train in analytical and logical skills than to consider, which subjects in mathematics could be more or less important in the later professional life.

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